Analytic wave solution with helicon and Trivelpiece-Gould modes in an annular plasma

Johan Carlsson*, Daniele Pavarin† and Mitchell Walker**

*Tech-X Corporation, Boulder, Colorado, USA
†University of Padua, Padova, Italy
**Georgia Institute of Technology, Atlanta, Georgia, USA

Abstract. Helicon sources in an annular configuration have applications for plasma thrusters. The theory of Klozenberg *et al.* [J.P. Klozenberg B. McNamara and P.C. Thonemann, J. Fluid Mech. **21** (1965) 545–563] for the propagation and absorption of helicon and Trivelpiece-Gould modes in a cylindrical plasma has been generalized for annular plasmas. Analytic solutions are found also in the annular case, but in the presence of both helicon and Trivelpiece-Gould modes, a heterogeneous linear system of equations must be solved to match the plasma and inner and outer vacuum solutions. The linear system can be ill-conditioned or even exactly singular, leading to a dispersion relation with a discrete set of discontinuities. The coefficients for the analytic solution are calculated by solving the linear system with singular-value decomposition.

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BACKGROUND

Klozenberg *et al.* [1] derived a wave equation with helicon and Trivelpiece-Gould solutions from Faraday's law

$$\nabla \times \mathbf{E} = i\omega \mathbf{B}\,,$$

pre-Maxwell Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\,,$$

and using the following form of Ohm's law to express plasma current $\mathbf{J}$ in terms of wave electric field $\mathbf{E}$,

$$i\omega \mathbf{J} = \boldsymbol{\omega}_{ce} \times \mathbf{J} - \omega_{pe}^2 \varepsilon_0 \mathbf{E}\,,$$

where ω is a complex frequency whose small imaginary part is a collision frequency. Solutions of the form $f(r)\exp[-i(\omega t - m\theta - kz)]$ are sought in cylindrical geometry.

The resulting wave equations is

$$\omega \nabla \times (\nabla \times \mathbf{B}) - k\omega_{ce} \nabla \times \mathbf{B} + \frac{\omega \omega_{pe}^2}{c^2} \mathbf{B} = 0$$

which can be written as

$$[(\nabla \times) - \beta_1][(\nabla \times) - \beta_2]\mathbf{B} = 0\,,$$

where

$$\beta_{1,2} = \frac{k\omega_{ce}}{2\omega}\left[1 \mp \sqrt{1 - 4\frac{\omega_{pe}^2\omega\Re(\omega)}{c^2k^2\omega_{ce}^2}}\right],$$

with index 1 for the helicon mode and index 2 for the Trivelpiece-Gould (TG) mode.

The general solution is the sum of the solutions to

$$\nabla^2\mathbf{B} + \beta_1^2\mathbf{B} = 0 \tag{1}$$

and

$$\nabla^2\mathbf{B} + \beta_2^2\mathbf{B} = 0\,, \tag{2}$$

respectively.

For a cylindrical plasma with radius b, one finds the solution for the axial component

$$B_z(r) = BJ_m(\gamma_1 r) + DJ_m(\gamma_2 r)\,,\ 0 < r < b\,,$$
$$B_z(r) = ikFK_m(kr)\,,\ r > b\,,$$

where $\gamma_{1,2}^2 = \beta_{1,2}^2 - k^2$ and J_m and K_m are Bessel functions. The azimuthal and radial components, B_θ and B_r, respectively, can be expressed in terms of B_z by use of Eqs. (1) and (2). By matching plasma and vacuum solutions at $r = b$, the three coefficients B, D and F can be eliminated. The condition that the global solution is unique leads to a dispersion relation.

SOLUTION IN ANNULAR PLASMA

In an annular plasma, with vacuum for $r < a$, plasma for $a < r < b$, and vacuum for $r > b$, the Klozenberg theory must be generalized. The general solution becomes

$$B_z(r) = ikAI_m(kr)\,,\ r < a\,,$$
$$B_z(r) = BJ_m(\gamma_1 r) + CY_m(\gamma_1 r) + DJ_m(\gamma_2 r) + EY_m(\gamma_2 r)\,,\ a < r < b\,,$$
$$B_z(r) = ikFK_m(kr)\,,\ r > b\,,$$

As before, Eqs. (1) and (2) can be used to find expressions for B_θ and B_r. With two plasma-vacuum interfaces and three magnetic-field components to match at each, we get a total of six matching equations for the six coefficients A, B, C, D, E and F. However, the amplitude of the general solution can not be found from the matching conditions and has to be imposed in some other way. We here choose to do this by setting $A = -i$. One could then use five of the six matching conditions to determine the remaining five coefficients.

Before finding the general solution, we will look at the pure helicon solution, where the coefficients $D = E = 0$. Matching $B_z(r)$ at $r = a$ and $r = b$, and B_θ at $r = a$, some calculation leads to

$$B = ikAI_m(ka)\frac{\frac{m}{\gamma_1 a}\frac{\beta}{k}Y_m(\gamma_1 a) + Y_m'(\gamma_1 a)}{J_m(\gamma_1 a)Y_m'(\gamma_1 a) - Y_m(\gamma_1 a)J_m'(\gamma_1 a)}\,,$$

$$C = -ikAI_m(ka)\frac{\frac{m}{\gamma_1 a}\frac{\beta}{k}J_m(\gamma_1 a) + J'_m(\gamma_1 a)}{J_m(\gamma_1 a)Y'_m(\gamma_1 a) - Y_m(\gamma_1 a)J'_m(\gamma_1 a)}$$

and

$$kF = -i\frac{BJ_m(\gamma_1 b) + CY_m(\gamma_1 b)}{K_m(kb)}\,.$$

We also get the dispersion relation

$$J_m(\gamma_1 a)Y'_m(\gamma_1 a) \neq Y_m(\gamma_1 a)J'_m(\gamma_1 a)\,,$$

which tells us for which values of the axial wave number k there is a unique solution.

This pure helicon solution is plotted in Fig. 1 using the parameters: m=1, a=5cm, b=12cm, k=6.21/m, $\omega = 3.77 \cdot 10^7 s^{-1}$, $\omega_{ce} = 8.80 \cdot 10^9 s^{-1}$ and $\omega_{pe} = 7.98 \cdot 10^{10} s^{-1}$.

Returning now to the general case with both helicon and Trivelpiece-Gould (TG) modes, we have five unknowns and six matching equations. We could choose five of the matching equations, but the 5×5 coefficient matrix will be singular for triplets of (ω, m, k) where the dispersion relation has a discontinuity. Using singular-value decomposition (SVD) not only allows us to handle singularities, but also lets us find the solution to the overdetermined system consisting of all six matching equations. More specifically, applying SVD to the system

$$\mathbf{Mx} = \mathbf{y}\,,$$

where $\mathbf{M}$ is the 6×5 coefficient matrix, $\mathbf{x} = (B, C, D, E, F)$ and
$\mathbf{y} = (ikAI_m(ka), imAI_m(ka)/a, kAI'_m(ka), 0, 0, 0)$, gives the least-squares fit of $\mathbf{x}$. The

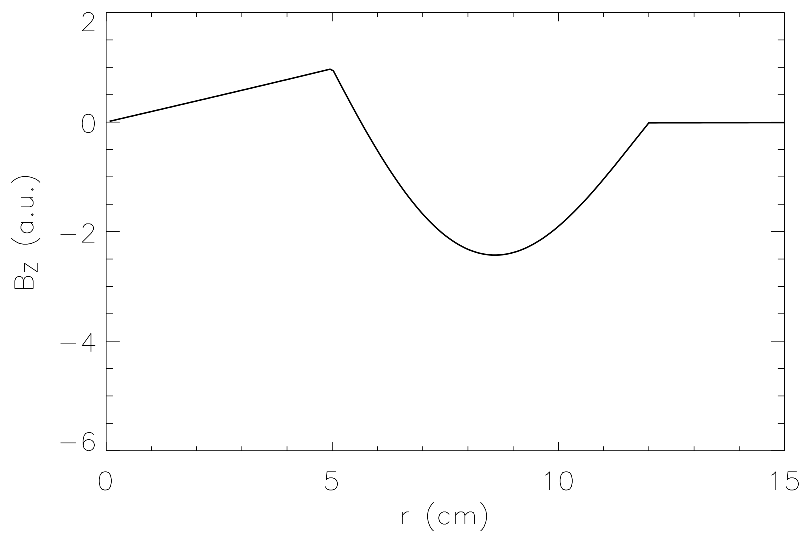

FIGURE 1. Axial magnetic field of pure helicon solution in annular plasma

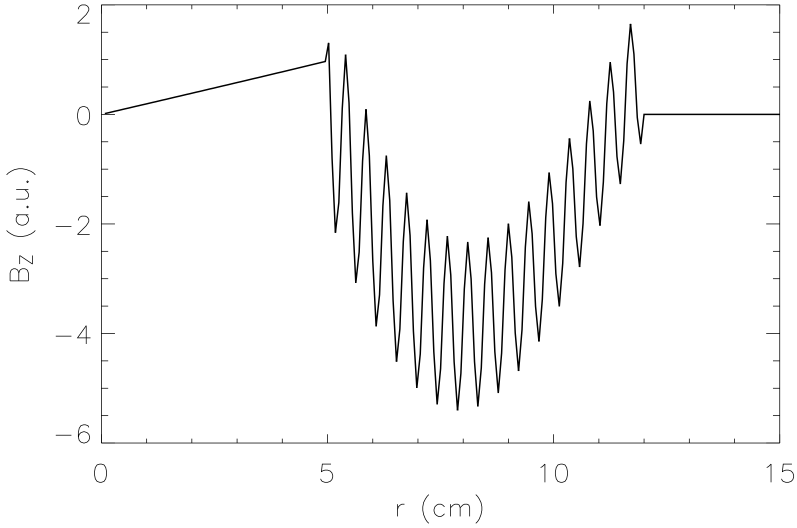

FIGURE 2. Axial magnetic field of solution with both helicon (long wavelength) and Trivelpiece-Gould (short wavelength) modes in annular plasma

matrix elements of $\mathbf{M}$ are given by

$$M_{1,1} = J_m(\gamma_1 a)\,,$$
$$M_{2,1} = -\gamma_1^{-2}[mkJ_m(\gamma_1 a)/a + \beta_1\gamma_1 J_m'(\gamma_1 a)]\,,$$
$$M_{3,1} = i\gamma_1^{-2}[m\beta_1 J_m(\gamma_1 a)/a + k\gamma_1 J_m'(\gamma_1 a)]\,.$$

$M_{4,1}, M_{5,1}, M_{6,1}$ are the same as $M_{1,1}, M_{2,1}, M_{3,1}$, but with $a \to b$. $M_{1,2}, M_{2,2}, M_{3,2}$ are the same as $M_{1,1}, M_{2,1}, M_{3,1}$, but with $J_m(\gamma_1 a) \to Y_m(\gamma_1 a)$ and $J_m'(\gamma_1 a) \to Y_m'(\gamma_1 a)$. $M_{4,2}, M_{5,2}, M_{6,2}$ are the same as $M_{1,2}, M_{2,2}, M_{3,2}$, but with $a \to b$. $M_{1,3}, M_{2,3}, M_{3,3}$ are the same as $M_{1,1}, M_{2,1}, M_{3,1}$, but with $\gamma_1 \to \gamma_2$. $M_{4,3}, M_{5,3}, M_{6,3}$ are the same as $M_{1,3}, M_{2,3}, M_{3,3}$, but with $a \to b$. $M_{1,4}, M_{2,4}, M_{3,4}$ are the same as $M_{1,2}, M_{2,2}, M_{3,2}$, but with $\gamma_1 \to \gamma_2$. $M_{4,4}, M_{5,4}, M_{6,4}$ are the same as $M_{1,4}, M_{2,4}, M_{3,4}$, but with $a \to b$. $M_{1,5} = M_{2,5} = M_{3,5} = 0$ and

$$M_{4,5} = -ikK_m(kb)\,,$$
$$M_{5,5} = -imK_m(kb)/b\,,$$
$$M_{6,5} = -kK_m'(kb)\,.$$

The coefficients are then calculated as $\mathbf{x} = \mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^H\mathbf{y}$, where the SVD of $\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$, where $\mathbf{V}^H$ is the conjugate transpose of $\mathbf{V}$, $\boldsymbol{\Sigma}$ is the diagonal matrix with the singular values as elements and $\boldsymbol{\Sigma}^+$ is its pseudoinverse. Using the same parameters as above, we get the solution shown in Fig. 2.

REFERENCES

1. J.P. Klozenberg B. McNamara and P.C. Thonemann, J. Fluid Mech. **21** (1965) 545–563.